

Recall: We wish to find a Yang connection in a Pr 1308
given $\mathbb{C}$ -orbit (= isom. class of hol. structures).

Definition (Mumford-Takemoto)

Slope $\mu(E) = \deg(E) / \text{rk}(E)$. A hol. structure E is stable
if for all proper holomorphic subbundles $S \hookrightarrow E$
there holds $\mu(S) < \mu(E)$ or, equivalently
 $\mu(E/S) > \mu(E)$

Equivalence: $\mu(E) = \frac{c_1(E)}{\text{rk}(E)} = \frac{c_1(S) + c_1(E/S)}{\text{rk}(S) + \text{rk}(E/S)}$

Motivation:

Holomorphic sub & quotient bundles

$S \hookrightarrow E$ is a hol. subbundle if the
map is hol. and injective + linear on fibers.

$\Rightarrow$ Can choose hol. frame $\{e_1, \dots, e_s, f_1, f_{s+1}, \dots, f_n\}$
frame for S not really a frame

$\Rightarrow$ hol. transition fns are of the form for E/S .

trans. fns
for S

$$\begin{pmatrix} g_S & * \\ 0 & g_{E/S} \end{pmatrix}$$

Note that these matrices also satisfy the cocycle condition

$\Rightarrow$ can define the quotient (hol.) bundle

$U = E/S$ to have trans. fns g_{ij} .

The underlying real bundle of U is just $S^\perp \subset E$.

$\rightarrow$ hol. map $E \rightarrow U$

Now let A be a unitary connection on E .

2

Working in a unitary frame $\{e_1, \dots, e_s, e_{s+1}, \dots, e_n\}$
 the connection form has the shape: $\underbrace{\text{unit. frame}}_{\text{for } S}$ $\underbrace{\text{u. frame for } E/S}$

$$A = \begin{pmatrix} A_S & \beta \\ -\beta^* & A_n \end{pmatrix} \quad (\text{matrix of 1-forms})$$

Here $\beta = \pi_S \circ D_A(e_{s+1}, \dots, e_n)$ is the 2nd fund. form.

If A is compatible with $\bar{\partial}_E$, then A_S

is compatible with $\bar{\partial}_S = \bar{\partial}_E|_S$ and A_n is

compat. with $\bar{\partial}_n = \pi_{S^\perp} \circ \bar{\partial}_E$.

Moreover, since S is a hol. subbundle, then

$$0 = \pi_{S^\perp} \bar{\partial}_E(e_1, \dots, e_s) = (-\beta^*)^{0,1}$$

$\Rightarrow \beta$ is a matrix of $(0,1)$ -forms.

$$\beta \in \Omega^{0,1}(S \otimes U^*), \quad \beta^* \in \Omega^{1,0}(U \otimes S^*)$$

Curvature:

$\text{Hom}(U, S)$

$\text{Hom}(S, U)$

$$\begin{aligned} F(A) &= dA + A \wedge A = \begin{pmatrix} dA_S & d\beta \\ -d\beta^* & dA_n \end{pmatrix} + \begin{pmatrix} A_S \wedge A_S - \beta \wedge \beta^* & A_S \wedge \beta + \beta \wedge A_n \\ -\beta^* \wedge A_S - A_n \wedge \beta^* & -\beta^* \wedge \beta + A_n \wedge A_n \end{pmatrix} \\ &= \begin{pmatrix} F(A_S) - \beta \wedge \beta^* & D_{S \otimes U} \beta \\ -D_{U \otimes S^*} \beta^* & F(A_n) - \beta^* \wedge \beta \end{pmatrix} \end{aligned}$$

On a Riemann surface: $\beta = \beta_{\bar{z}} d\bar{z}$ $(dx + i dy) = d\bar{z}$

$$\beta = \beta_{\bar{z}} d\bar{z} \quad i \cdot (-\beta \wedge \beta^*) = -i \beta_{\bar{z}} (\beta_{\bar{z}})^* d\bar{z} \wedge dz$$

$$\beta^* = (\beta_{\bar{z}})^* dz = -i \beta_{\bar{z}} \beta_{\bar{z}}^* \cdot 2i \cdot dx \wedge dy$$

$$F_{A_S} = +2 \beta_{\bar{z}} \beta_{\bar{z}}^* dx \wedge dy$$

$$\Rightarrow i \cdot F(A) \Big|_S = i \cdot F(A_S) - i \rho \wedge \rho^* \geq i F(A_S)$$

(rel. vol. form)

Curvature decreases in subbundles

$$i F(A) \Big|_S \geq i F(A_S) \quad \text{ptwise (rel. dV)}$$

increases in quotients.

$$i F(A) \Big|_M \leq i F(A_M)$$

Assume $\mathcal{E}$ is indecomposable for any

$0 \rightarrow S \rightarrow \mathcal{E} \rightarrow \mathcal{E}/S \rightarrow 0$, there does not exist

a holomorphic splitting map $\mathcal{E}/S \rightarrow \mathcal{E}$.

Means can't put the frame back in.

(would make $\mathcal{E} = S \oplus \mathcal{E}/S$ as hol. bundle.)

also assume $\text{Vol}(M) = 1$.

Theorem (N-S) $\mathcal{E}$ is stable $\Leftrightarrow \exists$ compat. conn. A

with $\forall F_A = -2\pi i \mu(\mathcal{E}) \cdot \text{Id.}$

(unique up to df .)

Donaldson's proof. For simplicity we assume W
 $\dim(E) = 2, c_1(E) = 0 = \mu(E)$.

So E is stable $\Leftrightarrow \deg(E(s)) > 0 \forall s \rightarrow E$ subbundle (thm.)

Fix $\langle \cdot, \cdot \rangle$ on E , vary $\bar{\partial}_E$ by g^E
 $\xrightarrow{\text{change}}$ connection $g(A)$ changes in a stable

Choose a minimizing sequence $g_i(A) = A_i$

for $Y_M = \frac{1}{2} \int |F_A|^2 dV$.

Lemma 0. (Karen Uhlenbeck) (1982) (Lang $M < 4$)

Given a sequence A_i of unitary connections
 on E with $\int |F_{A_i}|^2 < E < \infty$. If i , there
 exists a subsequence $\{i'\} \subset \{i\}$ and a set of

L^2 unitary gauge transformations $u_{i'} \in G(M)$,
 such that

$$u_{i'}(A_{i'}) \rightarrow B$$

weakly in L^2 , for some (L^2) unitary connection B on E .

(Will assume B is smooth.)
 $D_A \rightarrow C^0 \rightarrow$ works on E .

Proof uses Coulomb gauge fixing: $D_A^* A = 0$ (elliptic)
 (clears sim. to next lemma.) $dA + \star A \wedge A = F_A$ (system)

Write F for the hol. bundle defined by $\bar{\partial}_B$.

Fukaya's lemma $\Rightarrow Y_M(B) \leq \inf_{g \in G} Y_M(g(A))$.

Lemma 1 (Main Lemma!) There exists a nonlinear holomorphic map $\mathcal{E} \xrightarrow{\alpha} \mathcal{F}$.

Proof: Consider $\bar{\partial}_{AB}: \Omega^0(E \otimes E^*) \rightarrow \Omega^{0,1}(E \otimes E^*)$

Recall: $\bar{\partial}_{AB} s = 0 \iff s \in \text{Hom}(E, F)$ holomorphic

Assume the claim fails: then let $\bar{\partial}_{AB} = \{0\}$

$\bar{\partial}_{AB}$ is elliptic! By the Fredholm alternative,

$\exists C$ s.t. $\|s\|_{L^2(M)} \leq C \|\bar{\partial}_{AB} s\|_{L^2(M)} \quad \forall s \in \mathcal{R}^0(E \otimes E^*)$

Sobolev: $\|s\|_{L^4} \leq C_1 \|\bar{\partial}_{AB} s\|_{L^2}$

Now consider the minimizing sequence $A_i \rightarrow B$ weakly in L^2 . Since $\dim M < 4$, $L^2 \hookrightarrow L^4$

$\Rightarrow A_i \rightarrow B$ strongly in L^4 .

Note that $(\bar{\partial}_{AB} - \bar{\partial}_{AA_i})s = (B - A_i) \cdot s$

$\Rightarrow \|(\bar{\partial}_{AB} - \bar{\partial}_{AA_i})s\| \leq C_2 \|A_i - B\|_{L^4} \|s\|_{L^4}$

Compute

$$\begin{aligned} \|s\|_{L^4} &\leq C_1 \|\bar{\partial}_{AB} s\|_{L^2} \leq C_1 \|\bar{\partial}_{AA_i} s + (\bar{\partial}_{AB} - \bar{\partial}_{AA_i})s\|_{L^2} \\ &\leq C_1 (\|\bar{\partial}_{AA_i} s\|_{L^2} + C_2 \|A_i - B\|_{L^4} \|s\|_{L^4}) \end{aligned}$$

$\Rightarrow (1 - C_1 C_2 \|A_i - B\|_{L^4}) \|s\|_{L^4} \leq C_1 \|\bar{\partial}_{AA_i} s\|_{L^2}$

Choose i suff. large that $\|A_i - B\|_{L^4} < \frac{1}{C_1 C_2}$. Then $\bar{\partial}_{AA_i} g_i = 0 \Rightarrow g_i = 0$ and put $s = g_i$.

Now we claim that: if E is stable then α is an isomorphism.

Two cases: (a) $E \xrightarrow{\alpha} F$ has full rank somewhere

$$\Leftrightarrow 1^{\otimes 2} E \rightarrow 1^{\otimes 2} F \text{ is not identically zero}$$

$$\text{But } \deg E = \deg 1^{\otimes 2} E = \deg 1^{\otimes 2} F$$

$$\Rightarrow \deg(1^{\otimes 2} E \otimes (1^{\otimes 2} F)^*) = 0$$

$\Rightarrow$ any hol. section is nonvanishing.

(follows from computation showing $\deg \mathcal{O}(x) = 1$.)

$\therefore E \rightarrow F$ has full rank everywhere! $\Rightarrow$ isomorphism.

(b) $E \xrightarrow{\alpha} F$ has full rank nowhere (but is non-zero.)

$$\begin{array}{ccccccc} 0 & \longrightarrow & P & \longrightarrow & Q & \longrightarrow & \mathcal{O} \longrightarrow 0 \\ & & \downarrow \text{ker}(E \rightarrow F) & & \downarrow & & \\ 0 & \longrightarrow & N & \longrightarrow & F & \longleftarrow & M \longrightarrow 0 \end{array}$$

Exact rows
 P, Q, N, M
are bundles.

$$\text{Stability} \Rightarrow \deg Q = d > 0$$

$$\deg(E) = 0 \Rightarrow \deg P = -d < 0$$

To

constant M : Let z_0 be a point where α has

rank zero:

$$\alpha(z) = \begin{pmatrix} \alpha_{11}(z) & \alpha_{12}(z) \\ \alpha_{21}(z) & \alpha_{22}(z) \end{pmatrix} = z^k \begin{pmatrix} \tilde{\alpha}_{11}(z) & \tilde{\alpha}_{12}(z) \\ \tilde{\alpha}_{21}(z) & \tilde{\alpha}_{22}(z) \end{pmatrix}$$

where $0 \leq k = \min_{i,j} (\text{ord}_{z_0}(\alpha_{ij})) \Rightarrow$ some $\tilde{\alpha}_{ij}(z_0) \neq 0$.

Then $\mathcal{F}(z)$ defines a holomorphic map $\mathcal{M} = \mathcal{Q} \otimes \mathcal{O}(z_0)^{\otimes k} \rightarrow \mathcal{F}$

which is injective on the fiber over z_0 .

$\Rightarrow \mathcal{M}$ is a subbundle of $\mathcal{F}$. Moreover,

$$\deg(\mathcal{M}) = \deg(\mathcal{Q}) + k \deg \mathcal{O}(z_0) = d + k > d > 0.$$

Let $\mathcal{N} = \mathcal{F}/\mathcal{M}$.

Lemma 2: $\gamma_{\mathcal{M}}(B) \geq 4\pi^2 d^2$.

Proof: $B = \begin{pmatrix} B_m & \beta \\ -\beta^* & B_n \end{pmatrix}$

$$F(B) = \begin{pmatrix} F(B_m) - \beta \wedge \beta^* & D\beta \\ -D\beta^* & F(A_n) - \beta^* \wedge \beta \end{pmatrix}$$

Write $|F(B_m)| = f(z)$.

$$\int |F(B_m) - \beta \wedge \beta^*|^2 = \int (f(z) + |\beta|^2(z))^2$$

CURVATURE BECOMES

MORE POSITIVE.

$$\geq \left(\int (f(z) + |\beta|^2(z)) \right)^2$$

$$= (2\pi \deg(\mathcal{M}) + \int |\beta|^2(z)) \geq 4\pi d$$

$$\begin{aligned} \gamma_{\mathcal{M}}(B) &= \frac{1}{2} \int |F(B)|^2 \geq \frac{1}{2} \int |F(B_m) - \beta \wedge \beta^*|^2 + \frac{1}{2} \int |F(A_n) - \beta^* \wedge \beta|^2 \\ &\geq 4\pi d^2. \end{aligned}$$

Lemma 3. In case (D), there exists a
connection $g(A)$ with $\gamma_m(g(A)) < 4\pi^2 d^2$.

Proof: By proposition from last class, there exist compatible connections strict inequality!

A_S on S , A_U on U with

$$*F(A_S) = +2\pi i d, \quad *F(A_U) = -2\pi i d. \quad \text{Let } \omega = \frac{i}{2} g \bar{z} dz \wedge d\bar{z} = \text{volume form.}$$

$$\Leftrightarrow F(A_S) = -2\pi i d\omega, \quad F(A_U) = -2\pi i d\omega$$

cancel
signs

We may change by g to achieve this:

$$F(g(A)) = \begin{pmatrix} 2\pi i d\omega - \beta \wedge \beta^* & D\beta \\ -D\beta^* & -2\pi i d\omega - \beta^* \wedge \beta \end{pmatrix}$$

Note that β still must be nonzero, else the extension would split.

$$\text{Recall: } \beta \in \Omega^{0,1}, \quad F(g(A)) \in \Omega^{2,0}.$$

$$\therefore \bar{\partial}\beta = 0, \quad D\beta = \bar{\partial}\beta.$$

Hodge theory for the $\bar{\partial}$ -operator (Dolbeault operator) says that $\bar{\partial}\beta = 0 \Rightarrow \exists \alpha \in T(S \otimes U^*)$ s.t.

$$\bar{\partial}^*(\beta + \bar{\partial}\alpha) = 0, \quad \text{i.e. } \beta + \bar{\partial}\alpha \text{ is harmonic.}$$

$$\Delta_{\bar{\partial}} = 0.$$

Choose the gauge change $g = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$

$$\Rightarrow \bar{\partial}_A \text{ changes by } \begin{pmatrix} 0 & \bar{\partial}_A \alpha \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \bar{\partial}_A \alpha \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \beta \mapsto \beta + \bar{\partial}\alpha$$

d. e. we choose the "harmonic representative" for $\Delta_{\bar{g}}$.

Note : $\bar{\partial}_A^* = -iX \circ \bar{\partial}_A$ on $\Delta_{\bar{g}}$ surf.

$$(d\langle u d\bar{z}, w \rangle = \langle \bar{\partial}(u d\bar{z}), w \rangle + \langle u d\bar{z}, \bar{\partial} w \rangle)$$

$$\text{So } 0 = \bar{\partial} \beta = \bar{\partial}^* \beta = 0 = \bar{\partial} \beta.$$

$$\Rightarrow D\beta = (\bar{\partial} + \bar{\partial}^*)\beta = 0. \quad [\text{hook off along } \text{const}]$$

bestly, we may use the gauge transf $g = \begin{pmatrix} t & \\ & 1 \end{pmatrix}$

$$\text{to change } \bar{\partial}_A \left(\log + \bar{\partial}_A g g^{-1} = -(A^{0,1} g + g A^{0,1}) g^{-1} \right)$$

$$= - \begin{bmatrix} A_s^{0,1} \beta & \\ & A_m^{0,1} \end{bmatrix} = \begin{bmatrix} t A_s^{0,1} t^{-1} t \beta & \\ & A_m^{0,1} \end{bmatrix}$$

$$= \begin{pmatrix} 0 & (t - 1)\beta \\ 0 & 0 \end{pmatrix}$$

$\therefore$ we replace β by $t\beta$ and A_t .

$$\Rightarrow F(A_t) = \begin{pmatrix} 2\pi i d\omega - t\beta \beta^* & 0 \\ 0 & -2\pi i d\omega - t\beta^* \beta \end{pmatrix}$$

$$i \cdot F(A_t) = \begin{pmatrix} -2\pi d\omega + t|\beta|^2 & 0 \\ 0 & 2\pi i d\omega - t|\beta|^2 \end{pmatrix}$$

$$\Rightarrow \text{For } t \text{ suff. small, } y_m(A(t)) = \frac{1}{2} \|F_{A_t}\|^2 < 4\pi^2 d^2 \quad \square.$$

Conclude: $\mathcal{E} \xrightarrow{\sim} \mathcal{F}$ is an isomorphism

$B = g(A)$ attains the minimum in $\mathcal{O}$.

Is this a ym connection?

Assume $A : \forall F \neq 0$. Note that $\ker(D_A^* D_A) = \ker(D_A D_A^*) = \ker(D_A)$ on $\mathcal{D}^0(E \otimes \mathbb{C})$ consists of constant maps. χ index = $\dots$
 Since $\int_M \langle F, F \rangle dV = \int_M \langle F, F \rangle dV = 0$, the equation

$$(\ast) \quad D_A^* D_A h = -i \ast F$$

has a self-adj. soln: $h \neq 0$. Let $g_t = e^{th}$, $A_t = g_t(A)$.
 $\Rightarrow \partial_t A = -\bar{\partial} h + \partial h = \text{self-adj.}$
 $= -i \ast D h \Rightarrow \partial_t A = -i \ast D h$

$$t > 0 \Rightarrow 0 < \int_M \langle F, F \rangle dV = \int_M \langle i D_A^* D_A h, \ast F \rangle dV$$

$$= \int_M \langle i D h, D \ast F \rangle dV$$

$$= \int_M \langle i \ast D h, \ast D \ast F \rangle dV$$

$$\langle D \ast F, D h \rangle = - \int_M \langle \partial_t A, D_A^* F_A \rangle dV$$

$$= - \partial_t \langle \gamma_{\text{YM}}(A) \rangle$$

$\Rightarrow \gamma_{\text{YM}}$ decreases $\times$

□

In this case, we conclude F is flat!!

$\mu = 0$,
 can choose $\{e_\pm\} = \{e_\pm\}$ (standard orthonormal basis).

$$\text{Let } \langle F \rangle = \omega \cdot F = \frac{1}{2} g^{ij} F_{ij} \cdot \beta$$

Theorem (Don-Uhl-Yau). Assume E is stable.
 1985-6.

Then E admits a unique (up to g) connection with

$$\langle F \rangle = -2\pi i \mu(E) \quad (\text{Higgs } \gamma\text{-YM}).$$